Descriptive set theory and geometrical paradoxes I

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UCLA

Banach Tarski and equidecompositions

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If a is an action of a group Γ on a set X, then $A, B \subseteq X$ are a-**equidecomposable** if there is a finite partition $A_0, \ldots, A_n$ of A and group elements $\gamma_0, \ldots, \gamma_n \in \Gamma$ so that $\gamma_0 A_0, \ldots, \gamma_n A_n$ is a partition of B.

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We say a is **paradoxical** if we can partition X into two sets A, B that are each a-equidecomposable with X.

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- Let B be the unit ball in $\mathbb{R}^3$. Then one can find two rotations of $B \setminus \{0\}$ which generate a copy of $\mathbb{F}_2$. By using a paradoxical decomposition of $\mathbb{F}_2$ in each orbit, one can show that the action of these two rotations on $B \setminus \{0\}$ is paradoxical. This is how one proves the Banach-Tarski paradox.

Amenability

Consider the action of $\mathbb{Z}$ on itself via left translation. Let U be a nonprincipal ultrafilter on ω, and define the measure μ on subsets of $\mathbb{Z}$ by

$$\mu(A) = \lim_{n \in U} \frac{|A \cap \{-n, \dots, n\}|}{|\{-n, \dots, n\}|}$$

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Theorem (Tarski)

An action a of a group Γ on a set X is not paradoxical if and only if it is **amenable**. *That is, it admits a finitely additive a-invariant measure on all subsets of X such that $\mu(X) = 1$.*

Matchings and equidecompositions

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Suppose a is an action of a group Γ on a set X. Then $A, B \subseteq X$ are a-equidecomposable if and only if there is a finite set $S \subseteq \Gamma$ so that there is a perfect matching of the graph on $A \sqcup B$ where there is an edge from $x \in A$ to $y \in B$ if $\gamma \cdot x = y$ for some $\gamma \in S$.

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One can use Hall's theorem to give a short proof of Tarski's result that an action is amenable iff it is not paradoxical.

Theorem (Hall, Rado)

A locally finite bipartite graph G with bipartition $\{B_0, B_1\}$ has a perfect matching iff for every finite subset F of B_0 or B_1,

$$|\mathrm{N}(F)| \geq |F|$$

where $\mathrm{N}(F)$ is the set of neighbors of F.

Descriptive graph combinatorics

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Our goal is to describe some recent applications of these ideas to geometrical paradoxes.

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An easy counterexample: let A be a single unit ball in $\mathbb{R}^3$ and B be two unit balls in $\mathbb{R}^3$. Let S be the set of isometries needed to equidecompose A and B in the Banach-Tarski paradox, and let G be the associated graph on $A \sqcup B$. Then G satisfies Hall's condition (since there is an equidecomposition). However, G has no Borel (or even Lebesgue measurable) perfect matching, since there is no Lebesgue measurable solution to the Banach-Tarski paradox.

Baire measurable paradoxes

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There is a paradoxical decomposition of the unit ball in $\mathbb{R}^3$ using pieces with the Baire property.

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Recently, we have a more general result:

Theorem (M.-Unger, 2015)

Suppose Γ is a group acting on a Polish space X by Borel automorphisms. If the action has a paradoxical decomposition, then it admits a paradoxical decomposition where each piece has the Baire property.

A Baire measurable matching lemma

Our proof uses the following Baire measurable version of Hall's theorem

Theorem (M.-Unger, 2015)

Suppose G is a locally finite bipartite Borel graph on a Polish space with bipartition $\{B_0, B_1\}$ and there exists an $\epsilon > 0$ such that for every finite set F with $F \subseteq B_0$ or $F \subseteq B_1$,

$$|\mathrm{N}(F)| \geq (1 + \epsilon)|F|.$$

Then there is a Baire measurable matching of G.

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Laczkovich's counterexample to Miller's question shows that we cannot improve ϵ to 0.

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Suppose $A, B \subseteq \mathbb{R}^3$ are bounded sets with nonempty interior and the same Lebesgue measure. Then A and B are equidecomposable using Lebesgue measurable pieces.

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Their proof uses a measurable version of Hall's matching theorem due to Lyons and Nazarov.

Tarski's circle squaring problem

The theory of amenability can be used to show that Lebesgue measure on $\mathbb{R}^2$ can be extended to a finitely additive isometry-invariant measure on $\mathbb{R}^2$. Hence, there is no version of the Banach-Tarski paradox in $\mathbb{R}^2$.

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Dubins, Hirsch, and Karush (1963) had shown that Tarski's circle squaring cannot be solved uses pieces whose boundaries consist of a single Jordan curve.

A Borel solution to Tarski's circle squaring problem

Theorem (M.-Unger, 2016)

Tarski's circle squaring problem can be solved using Borel pieces. More generally, suppose $k \geq 1$ and $A, B \subseteq \mathbb{R}^k$ are bounded Borel sets such that $\lambda(A) = \lambda(B) > 0$, $\Delta(\partial A) < k$, and $\Delta(\partial B) < k$. Then A and B are equidecomposable by translations using Borel pieces.

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Here λ is Lebesgue measure, ∂A is the boundary of A, and Δ is upper Minkowski dimension.

This is a Borel version of a general equidecomposition theorem originally due to Laczkovich (1992). Grabowski, Máthé, and Pikhurko had proved a measurable/Baire measurable version in 2015.

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Our equidecomposition of the circle and square uses $\approx 10^{200}$ pieces which are finite boolean combinations of Σ^0_4 sets.

In the remaining two lectures, we sketch the proof of this theorem. The proof heavily uses ideas from the study of flows in networks, and also recent work of Gao, Jackson, Krohne and Seward on special witnesses to the hyperfiniteness of Borel actions of $\mathbb{Z}^d$.